Z-Equilibrium For a CSP Game

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Abstract

Constraint programming and game theory are two active research domains providing a powerful framework for modeling and solving several important applications in computer science, artificial intelligence and decision aiding in general. However, only little attention has been paid to their possible connexions and relationships. In this paper, we first prove the equivalence between the concept of solution of a constraint satisfaction problem and the Z-equilibrium of its associated game. Then, we propose a backtrack search based procedure for computing such equilibrium.

Introduction

Constraint programming (CP) is a flexible programming paradigm in which problems are specified declaratively as constraints between variables and a general search procedure is responsible for finding a valuation of the variables satisfying the constraints. Constraint Programming draws on a wide range of methods from artificial intelligence, computer science, databases, logic programming and operations research. It has been successfully applied in a number of fields such as scheduling, configuration, computational biology and vehicle routing.

On the other hand, game theory is a branch of mathematics devoted to studying interactions among rational agents. It can be used as framework for modeling and solving a variety of problems encountered in several application domains such as economy, transportation and logistics, telecommunications and biology.

The rise of distributed/parallel computing made game theory increasingly necessary for computer scientists to study settings in which intelligent agents reason and interact with other agents. One can cite, distributed constraint satisfaction problems where the problem is modeled as a constraint network in which variables and constraints are distributed among multiple agents (Yokoo et al. 1992). Various applications of Distributed Artificial Intelligence (DAI) can be formalized as distributed CSPs (Béjar et al. 2005; Yokoo and Hirayama 2000). In this dynamic, there are some innovative work establishing relationships between game theory and constraint satisfaction problems (CSP). Indeed, (Ricci 1991) proposed a representation of a CSP as a non-cooperative game with n players. He proved that each solution of a given CSP is also a Nash equilibrium of the associated game. Furthermore, when the CSP is consistent, the set of solutions of the CSP coincides with the set of admissible Nash equilibria of the game. Another possible modeling can be obtained by representing CSPs as cooperative games (Bistarelli and Gosti 2009), aiming to solve a distributed CSP via naming game. Furthermore, in (Kolaitis and Vardi 2000), the authors proved that the main consistency concepts used to derive tractability results for CSPs are intimately related to certain combinatorial pebble games, called the existential k-pebble games. More precisely, they particularly show that strong k-consistency is equivalent to a condition on winning strategies for the duplicator player in the existential k-pebble game.

We can also represent a game as a constraint satisfaction problem solving. Bordeaux and Pajot adopted this approach for computing Nash equilibrium (Bordeaux and Pajot 2005). For the same purpose, simple backtrack search methods have been adopted in (Porter, Nudelman, and Shoham 2008). In (Vickrey and Koller 2002; Soni, Singh, and Wellman 2007), efficient algorithms based on constraint satisfaction were proposed for computing approximate Nash equilibria for arbitrary one-shot graphical games. A similar approche was also proposed in (Soni, Singh, and Wellman 2007) for solving repeated graphical games. In (Apt, Rossi, and Venable 2008), the authors have compared the notions of optimality in strategic game and soft constraints. It is important to note that most of the contributions in this domain focused on Nash equilibrium, a fundamental concept in game theory (Jiang 2007; Porter, Nudelman, and Shoham 2008; Bordeaux and Pajot 2005).

In this paper, we deal with another related concept called Z-equilibrium, and we prove its equivalence with a solution of its associated CSP. As no algorithm exists in the literature allowing us to compute Z-equilibrium, we exploit this theoretical results to propose a backtrack search procedure for computing a Z-equilibrium of the CSP game.

Preliminary definitions and notations

The class of problems that constraint programming systems focus on are constraint satisfaction problems. A Constraint Network (CN) $\mathcal{P}$ is a pair $(\mathcal{X}, \mathcal{C})$ where $\mathcal{X}$ is a finite set of n variables and $\mathcal{C}$ a finite set of e constraints. Each variable $X \in \mathcal{X}$ has an associated domain, denoted D_X, which contains the set of values allowed for X. Each constraint $C \in \mathcal{C}$ involves an ordered subset of variables of $\mathcal{X}$, called scope and denoted $scp(C)$, and has an associated relation, denoted R_C, which contains the set of tuples allowed for its variables. The arity of the constraint C is given by $k_C = |scp(C)|$. A solution to a CN is the assignment of a value to each variable such that all the constraints are satisfied. A CN is said to be satisfiable (or consistent) iff it admits at least one solution. Constraint Satisfaction Problem (CSP) consists in determining whether a given CN is satisfiable or not. A CSP instance is then defined by a CN, and solving it involves either finding one solution or proving its unsatisfiability.

Representing a CSP as a Game

Solving a CSP may be reduced to the problem of finding an equilibrium in a game (Ricci 1991; Bistarelli and Gosti 2009; Kolaitis and Vardi 2000; Apt, Rossi, and Venable 2008).

Let us consider a constraint network $\mathcal{P} = (\mathcal{X}, \mathcal{C})$ and its associated game $\mathcal{G}(\mathcal{P})$ (Ricci 1991):

$$\mathcal{G}(\mathcal{P}) \quad = \quad < I, \{S_i\}_{i \in I}, \{U_i\}_{i \in I} >, \qquad (1)$$

- Each variable $X_i \in \mathcal{X}$ is associated to a player i. We note by $I = \{1, \ldots, n\}$ the set of players involved in the game $\mathcal{G}(\mathcal{P})$. There is a one to one mapping between the set of variables $\mathcal{X}$ and the set of players I.
- $S_i = D_{X_i}$ is the set of pure strategies associated to the player $i \in I$ i.e. there is a one to one mapping between the values of the domain of the variable X_i and the pure strategies of the player i.
- The utility function of the player $i \in I$ is given by

  $$U_i(s_1, \ldots, s_n) = \sum_{C_j \in \mathcal{C}(X_i)} k_{C_j} \times \chi_{C_j}(s_{j_1}, \ldots, s_{j_{k_{C_j}}}) \quad (2)$$

  where $s = (s_1, \ldots, s_n) \in S = \prod_{i=1}^{n} S_i$ is a complete instantiation of the n variables of the CSP $\mathcal{P}$ corresponding to a game issue (situation) of $\mathcal{G}(\mathcal{P})$ and $\mathcal{C}(X_i)$ defines the set of constraints involving the variable X_i. For a constraint $C_j \in \mathcal{C}(X_i)$ with $scp(C_j) = \{X_{j_1}, \ldots, X_{j_{k_{C_j}}}\}$ and a game issue s, we note $(s_{j_1}, \ldots, s_{j_{k_{C_j}}})$ the projection of s on the players $\{j_1, \ldots, j_{k_{C_j}}\} \subseteq I$. We define,

  $$\chi_{C_j}(s_{j_1}, \ldots, s_{j_{k_{C_j}}}) = \begin{cases} 1, & \text{if } (s_{j_1}, \ldots, s_{j_{k_{C_j}}}) \in R_{C_j}, \\ 0, & \text{otherwise.} \end{cases}$$

The utility or payoff function of a player $i \in I$ measures the degree of satisfaction of the constraints involving X_i. The payoff represents the number of constraints involving X_i that the game issue satisfies, weighted by the arity of the constraints.

As mentioned in (Ricci 1991), to express the relevance or the hierarchy of the constraints, one could associate a different weight for every constraint. Other weighting schemes can also be defined by associating different weights on variables or players.

In the sequel, we heavily use the following notations and definitions:

1. $s_i \in S_i$: the strategy chosen by a player $i \in I$;
2. s_{-i}: a combination of strategies chosen by the opponents of a player $i \in I$;
3. $s = (s_i, s_{-i})$: an issue (situation) of the game;
4. $S_{-i} = \prod_{j=1, j \neq i}^{n} S_j$: the set of all combinations of strategies that can be chosen by the opponents of a player $i \in I$;
5. $S = S_i \times S_{-i} = \prod_{i=1}^{n} S_i$: the set of all the issues of the game.

Strategy of Security and Security Gain

A player who doesn't like taking risks could choose to implement a secure strategy (also called maximin strategy) which in the worst case scenarios, assures him a minimum utility value, called security gain. It represents the largest value that the player can be sure to get without knowing the strategies of the other players.

By definition, we say that a strategy $s_i^* \in S_i$ is a security strategy for a player $i \in I$, if:

$$s_i^* \in \arg\max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} U_i(s_i, s_{-i}). \qquad (3)$$

The value of the expression $\max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} U_i(s_i, s_{-i})$ is the gain of security of the player $i \in I$.

Equilibrium Concept

An equilibrium is an issue where each player has no interest to update its behavior when the behavior of the other players is known. Different equilibrium concepts are defined in game theory.

Definition 1 (Nash Equilibrium) *An issue $s^* = (s_i^*, s_{-i}^*) \in S$ is a Nash equilibrium (in pure strategies) of the game $\mathcal{G}(\mathcal{P})$, if:*

$$U_i(s_i^*, s_{-i}^*) \geq U_i(s_i, s_{-i}^*), \qquad \forall s_i \in S_i, \; \forall i \in I.$$

Admissible Nash equilibrium is a refinement of the concept of Nash equilibrium.

Definition 2 *Let $s, s' \in S$ be two issues of the game $\mathcal{G}(\mathcal{P})$. We say that an issue $s \in S$* dominates *$s' \in S$, if they satisfy the following system of inequalities:*

$$U_i(s) \geq U_i(s'), \qquad \forall i \in I,$$

where, at least, one of them is strictly satisfied (>).

Definition 3 (Admissible Nash equilibrium) *A Nash equilibrium is said to be* admissible*, if it is not dominated by any other Nash equilibrium.*

Using Pareto optimality, we introduce another refinement of the Nash equilibrium.

Definition 4 (Pareto-optimal Nash equilibrium) *An issue $s^* \in S$ is a Pareto-optimal Nash equilibrium of the game $\mathcal{G}(\mathcal{P})$, if*

(a) $s^* \in S$ is a Nash equilibrium of the game $\mathcal{G}(\mathcal{P})$; and

(b) another issue $s \in S$ dominating $s^* \in S$ doesn't exist.

The concept of Z-equilibrium is introduced by Zhukovskii et al. (Zhukovskii and Chikrii 1994; Vaisbord and Zhukovskii 1980) in deterministic differential games and in two-player stochastic differential games by Gaidov in (Gaidov 1975). The Z-equilibrium is based on the concept of Pareto-optimality (Gaidov 1984; 1986b) and represents further development in the theory in comparison with the Nash-equilibrium (Gaidov 1986a).

Definition 5 (Z-equilibrium) *An issue $s^* \in S$ is a Z-equilibrium of the game $\mathcal{G}(\mathcal{P})$, if*
(a) *s^* is an active equilibrium, i.e. $\forall i \in I$, $\forall s_i \in S_i$, $s_i \neq s_i^*$, $\exists t_{-i} \in S_{-i}$ such that $U_i(s_i, t_{-i}) \leq U_i(s^*)$; and*
(b) *$s^* \in S$ is Pareto optimal, i.e. another issue $s \in S$ dominating s^* doesn't exist.*

The following properties of the Z-equilibrimu motivates our choice for it as the most suitable concept of solution for the game $\mathcal{G}(\mathcal{P})$:

Property 1 *(Ferhat and Radjef 2008) Let $s^* = (s_i^*, s_{-i}^*)$ be a Z-equilibrium of the game $\mathcal{G}(\mathcal{P})$.*

(a) The condition stating that s^* is an active equilibrium guarantees its stability.

(b) Z-equilibrium is individually and collectively rational.

(c) As s^* is Pareto optimal, it allows us to avoid the Tucker paradox.

Indeed, as a Z-equilibrium is an active equilibrium, its stability is guaranteed because in such situation, any deviation strategy of a player $i \in I$ generates a reaction of his opponents that decreases his own gain. This reduction is not desired by any rational player, so the stability of the situation of the game is guaranteed.

Note that in game theory, rationality is a main hypothesis which states that each player participating in the game adopts strategies leading to an increase in its gain. Individual rationality of Z-equilibrium is justified by the fact that such an outcome guarantees to each player, a gain at least equal to its security gain. Since this equilibrium is Pareto optimal, it provides both the collective interest, and avoids the Tucker paradox where many equilibriums may exist and the choice of one of them is not always obvious (Poundstone 1993).

From the definition of the utility function given in relation (2), we deduce the following proposition:

Proposition 1 *If $s^* \in S$ is a solution of the CSP $\mathcal{P}$, then*

$$U_i(s^*) = \max_{s \in S} U_i(s), \ \ \forall i \in I. \tag{4}$$

***Remark* 1** *The relation (4) has been established in (Ricci 1991) while considering sets of mixed strategies, It remains true for sets of pure strategies. Indeed, if s^* is a solution of the CSP $\mathcal{P}$, this means that s^* satisfies all the constraints of the set $\mathcal{C}$. Hence, $\chi_{C_j}(s_{j_1}^*, \ldots, s_{j_{k_{C_j}}}^*) = 1$, $\forall C_j \in \mathcal{C}(X_i)$, $\forall i \in I$. This implies that $U_i(s^*) = \sum_{C_j \in \mathcal{C}(X_i)} k_{C_j} = \max_{s \in S} U_i(s)$, $\forall i \in I$.*

The following theorem gives us the conditions of the existence of a Z-equilibrium in pure strategies for a finite normal game.

Theorem 1 *The game $\mathcal{G}(\mathcal{P})$ defined in relation (1) admits a Z-equilibrium in pure strategies.*

***Proof* 1** *The game $\mathcal{G}(\mathcal{P})$ associated to the CSP $\mathcal{P}$ is finite, since the set of strategies S_i for each player $i \in I$ is non empty and finite, keeping in mind that S_i corresponds to the domain D_{X_i} of the variable X_i in the CSP $\mathcal{P}$.*

For a finite game, the security gain

$$\alpha_i = \max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} U_i(s_i, s_{-i})$$

exists for any player $i \in I$.

Let

$$A = \{s \in S, \ \ U_i(s) \geq \alpha_i, \ \ \forall i \in I\}, \tag{5}$$

be the set of all issues of the game for which each player gets an utility at least equal to the security gain.

To prove that A is a non empty set, we note by s_i^G the security strategy of the player $i \in I$, defined as follows:

$$\alpha_i = \max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} U_i(s_i, s_{-i}) = \min_{s_{-i} \in S_{-i}} U_i(s_i^G, s_{-i}).$$

Let us consider the game issue generated from the choice by each player's security strategy i.e. $s^G = (s_1^G, \ldots, s_n^G) \in S$ with $s_i^G \in S_i$, $\forall i \in I$.

We have:

$$U_i(s^G) = U_i(s_i^G, s_{-i}^G) \geq \min_{s_{-i} \in S_{-i}} U_i(s_i^G, s_{-i});$$

$$\min_{s_{-i} \in S_{-i}} U_i(s_i^G, s_{-i}) = \max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} U_i(s_i, s_{-i}) = \alpha_i.$$

Hence,

$$U_i(s^G) \geq \alpha_i, \ \ \forall i \in I,$$

and $A \neq \emptyset$.

Fixing a n-vector $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_i \in]0,1[$, $\forall i \in I$, let us compute

$$\max_{s\in A} \sum_{i=1}^{n} \lambda_i U_i(s) = \sum_{i=1}^{n} \lambda_i U_i(s^*). \qquad (6)$$

The max *bound is reached on the point $s^* \in A$, because A is a finite set.*

Let us show that s^* is a Z-equilibruim:

We suppose that the player $i \in I$ updates its strategy $s_i \in S_i$. We set

$$t_{-i} = \arg \min_{v_{-i}\in S_{-i}} U_i(s_i, v_{-i})$$

the strategy of the remaining players in response to strategy updates of the player $i \in I$. We get:

$$U_i(s_i, t_{-i}) = \min_{v_{-i}\in S_{-i}} U_i(s_i, v_{-i}) \leq \max_{s_i\in S_i} \min_{v_{-i}\in S_{-i}} U_i(s_i, v_{-i}),$$

$$\max_{s_i\in S_i} \min_{v_{-i}\in S_{-i}} U_i(s_i, v_{-i}) = \alpha_i \leq U_i(s^*),$$

this prove that the issue s^* is an active equilibrium.

Let now prove that s^* is Pareto optimal. We suppose the contrary, i.e. there exists another issue $\widetilde{s} \in S$ which verify the system of inequalities

$$U_i(\widetilde{s}) \geq U_i(s^*); \;\; \forall i \in I$$

where, at least one, is strictely verified. By multiplying each of these inequalities with $\lambda_i \in]0,1[$, $i \in I$ and computing their sum, we deduce:

$$\sum_{i=1}^{n} \lambda_i U_i(\widetilde{s}) > \sum_{i=1}^{n} \lambda_i U_i(s^*),$$

which contradicts the relation (6) and prove that s^* is Pareto optimal. As we proved that s^* is an active equilibrium, then it is also a Z-equilibruim.

We have the following relations between the different equilibria of $\mathcal{G}(\mathcal{P})$:

Proposition 2
Any Pareto-optimal Nash equilibrium is both an admissible Nash equilibrium and a Z-equilibrium.

***Proof* 2** *1. Let $s^* \in S$ be a Pareto optimal Nash equilibrium. Obviously, it is a Nash equilibrium. Let us prove that it is also admissible.*

Let $\widetilde{s} \in S$ another Nash equilibrium. As s^* is Pareto optimal, then s^* and $\widetilde{s} \in S$ can not verify the following system of inequalities

$$U_i(\widetilde{s}) \geq U_i(s^*); \;\; \forall i \in I$$

where, at least, one of them is strictly satisfied. This mean that s^* is admissible.

2. Let us show that if $s^* \in S$ is a Pareto optimal Nash equilibrium, then s^* is a Z-equilibrium.

Let s^* be a Pareto optimal Nash equilibrium. We have by definition:

$$\forall i \in I, \;\; \forall s_i \in S_i, \;\; U_i(s_i, s^*_{-i}) \leq U_i(s^*).$$

Hence, for each strategy updates $s_i \in S_i$ by any player $i \in I$, it is enough for the remaining players to maintain their strategies s^*_{-i} of the Nash equilibrium, to obtain

$$U_i(s_i, s^*_{-i}) \leq U_i(s^*),$$

we deduce that Nash equilibrium is an active equilibrium. Additionally, as s^* is Pareto optimal, we conclude that s^* is a Z-equilibrium.

Equivalence Between a Solution of a CSP $\mathcal{P}$ and a Z-equilibrium of $\mathcal{G}(\mathcal{P})$

Let us consider a CSP $\mathcal{P}$, and its associated game $\mathcal{G}(\mathcal{P})$. The following results establish the relationships between the solutions of the two problems.

Proposition 3 *Each solution of the CSP $\mathcal{P}$ is a Z-equilibrium of its associated game $\mathcal{G}(\mathcal{P})$.*

***Proof* 3** *Let $s^*=(s^*_1, \ldots, s^*_n)=(s^*_i, s^*_{-i})\in S$ be a solution of the CSP $\mathcal{P}$ and $\mathcal{G}(\mathcal{P})$ its associated game.*

From Proposition 1, we have:

$$U_i(s^*) = \max_{s\in S} U_i(s), \;\; \forall i \in I. \qquad (7)$$

If a player $i \in I$ changes his strategy s^*_i by $s_i \in S_i$, $s_i \neq s^*_i$ then it's sufficient that the remaining players maintain their strategy s^*_{-i} of s^* to have:

$$U_i(s^*) = \max_{s\in S} U_i(s) \geq U_i(s_i, s^*_{-i}),$$

From where we deduce that s^* is an active equilibrium of the game $\mathcal{G}(\mathcal{P})$.

Let us proof that s^* is Pareto optimal. Suppose the contrary, i.e there exists $\widetilde{s} \in S$ verifying the following system of inequalities

$$U_i(\widetilde{s}) \geq U_i(s^*); \;\; \forall i \in I \qquad (8)$$

and for at least one indice $j \in I$, the following relation holds,

$$U_j(\widetilde{s}) > U_j(s^*). \qquad (9)$$

The relation (9) contradicts the relation (7). Consequently, s^* is Pareto-optimal and we conclude that s^* is a Z-equilibrium of the game $\mathcal{G}(\mathcal{P})$.

Proposition 4 *If the set of solution of the CSP $\mathcal{P}$ is not empty, then any Z-equilibrium of its associated game corresponds to a solution of $\mathcal{P}$.*

***Proof* 4** *Let $s^* \in S$ be a Z-equilibrium of the game $\mathcal{G}(\mathcal{P})$ associated to a CSP $\mathcal{P}$. Suppose that s^* is not a solution of the CSP $\mathcal{P}$. As the set of solutions of the CSP $\mathcal{P}$ is not empty, then we set $\tilde{s}$ one of its solutions.*

From Proposition 1, $\tilde{s}$ verifies the following relation:

$$U_i(\tilde{s}) = \max_{s \in S} U_i(s), \qquad \forall i \in I. \tag{10}$$

As we assumed that s^* is not a solution of $\mathcal{P}$, this means that s^* does not satisfy some constraints. Let us note by $\overline{\mathcal{C}}(s^*)$ the set of constraints not satisfied by s^* and consider the set $\overline{\mathcal{X}}(s^*)$ of the variables involved in the constraints $C \in \overline{\mathcal{C}}(s^*)$. Let $i \in \overline{\mathcal{X}}(s^*)$. We have:

$$U_i(\tilde{s}) = U_i(\tilde{s}_1, \ldots, \tilde{s}_n) = \sum_{C_j \in \mathcal{C}(X_i)} k_{C_j} \chi_{C_j}(\tilde{s}_{j_1}, \ldots, \tilde{s}_{j_{k_{C_j}}})$$

$$= \sum_{C_j \in \mathcal{C}(X_i) \backslash \overline{\mathcal{C}}(s^*)} k_{C_j} \overbrace{\chi_{C_j}(\tilde{s}_{j_1}, \ldots, \tilde{s}_{j_{k_{C_j}}})}^{1} + \sum_{C_j \in \overline{\mathcal{C}}(s^*) \cap \mathcal{C}(X_i)} k_{C_j} \overbrace{\chi_{C_j}(\tilde{s}_{j_1}, \ldots, \tilde{s}_{j_{k_{C_j}}})}^{1}$$

$$U_i(\tilde{s}) > \sum_{C_j \in \mathcal{C}(X_i) \backslash \overline{\mathcal{C}}(s^*)} k_{C_j} \chi_{C_j}(s^*_{j_1}, \ldots, s^*_{j_{k_{C_j}}}) + \sum_{C_j \in \overline{\mathcal{C}}(s^*) \cap \mathcal{C}(X_i)} k_{C_j} \overbrace{\chi_{C_j}(s^*_{j_1}, \ldots, s^*_{j_{k_{C_j}}})}^{=0}$$

$$U_i(\tilde{s}) = U_i(s^*).$$

Thus,

$$\forall i \in \overline{\mathcal{X}}(s^*), \;\; U_i(s^*) < U_i(\tilde{s}) = \max_{s \in S} U_i(s). \tag{11}$$

In addition, as s^* is a Z-equilibrium, then it is also Pareto optimal, i.e for $\tilde{s} \in S$ we have either

$$U_i(s^*) = U_i(\tilde{s}), \qquad \forall i \in I; \tag{12}$$

or there exists $k \in I$ such that

$$U_k(\tilde{s}) < U_k(s^*). \tag{13}$$

The relation (12) contradicts (11). Similarly, the relation (13) contradicts (10). This terminates the proof.

Computing Z-equilibrium

The procedure we propose for computing Z-equilibrium is based on backtrack search and on the proof of Theorem 1 stating the existence of a Z-equilibrium $s^* \in S$ in the game $\mathcal{G}(\mathcal{P})$. The steps of this procedure are described as follows:

1. Define the constraint network $\mathcal{P}$=$(\mathcal{X}, \mathcal{C})$ to solve, then consider its associated game $\mathcal{G}(\mathcal{P})$;
2. Find all possible issues of $\mathcal{G}(\mathcal{P})$, by instantiating sequentially the variables $\mathcal{X}$ of $\mathcal{P}$. Then, store them in a set S;
3. Evaluate each element of S for all players using the utility function defined by the relation (2). Store the evaluations in a matrix G, where each row represents the gains of a player in all the situations of the game;
4. Determine from the matrix G:
   (a) The security gain for each player $i \in I$;
   (b) The set A defined in relation (5) of all issues that guaranty for all the players an utility that is at least equal to security gain.
5. Fix a n-vector λ with $\lambda_j \in]0, 1[$, then evaluate each element $x \in A$, by the function:

$$f(x, \lambda) = \sum_{j=1}^{n} \lambda_j U_j(x), \;\; x \in A;$$

6. The Z-equilibrium is the issue

$$x^* = \arg(\max_{x \in A} f(x, \lambda)),$$

it corresponds to a solution of the constraint network $\mathcal{P}$.

Conclusion

In this paper, we formulated a constraint satisfaction problem as a noncooperative game with n-players. We established the relationship between the solutions of a constraint satisfaction problem and the Z-equilibrium of its associated game. Then, a new backtrack search based procedure for computing Z-equilibrium is proposed. This work opens several interesting perspectives. The implementation and the experimental validation of our procedure is a short term issue. We also plan to study the characteristics of the solutions obtained by our procedure comparatively to the classical solutions of the CSP.

References

Apt, K. R.; Rossi, F.; and Venable, K. B. 2008. Comparing the notions of optimality in strategic games and soft constraints. *Annals of Mathematics and ARtificial Intelligence* 52:25–54.

Béjar, R.; Domshlak, C.; Fernández, C.; Gomes, C.; Krishnamachari, B.; Selman, B.; and Valls, M. 2005. Sensor networks and distributed CSP: Communication, computation and complexity. *Artificial Intelligence* 161(1-2):117–147.

Bistarelli, S., and Gosti, G. 2009. Solving CSPs with naming games. In *Recent Advances in Constraints*, 16–32. Springer.

Bordeaux, L., and Pajot, B. 2005. Computing equilibria using interval constraints. In *Recent Advances in Constraints*, 157–171. Springer.

Ferhat, A., and Radjef, M. S. 2008. Existence conditions of a Zm-equilibrium for multicriteria games. In *13th International Symposium on Dynamic Games and Applications*.

Gaidov, S. D. 1975. *Deterministic and Stochastic Optimal Control*. Verlag, Berlin-Heidlberg-New York.

Gaidov, S. D. 1984. Basic optimal strategies in stochastic differential games. *C.R. Bulgarian Academy of Science* 37:457–460.

Gaidov, S. D. 1986a. Nash equilibrium in stochastic differential games. *Computers and Mathematics with Applications* 12A:761–768.

Gaidov, S. D. 1986b. Pareto-optimality in stochastic differential games. *Problems of Control and Information Theory* 15:439–450.

Jiang, M. 2007. Finding pure nash equilibrium of graphical game via constraints satisfaction approach. In *Combinatorics, Algorithms, Probabilistic and Experimental Methodologies*, 483–494. Springer.

Kolaitis, P. G., and Vardi, M. Y. 2000. A game-theoretic approach to constraint satisfaction. In *Proceedings of the Seventeenth National Conference on Artificial Intelligence (AAAI'00)*, 175–181.

Porter, R.; Nudelman, E.; and Shoham, Y. 2008. Simple search methods for finding a nash equilibrium. *Games and Economic Behavior* 63(2):642–662.

Poundstone, W. 1993. *Prisoner's Dilemma : John von Neumann, Game Theory, and the Puzzle of the Bomb*. Oxford University Press, Oxford, U.

Ricci, F. 1991. Equilibrium theory and constraint networks. In *Proceedings of and the International Conference on Game Theory*.

Soni, V.; Singh, S.; and Wellman, M. P. 2007. Constraint satisfaction algorithms for graphical games. In *Proceedings of the 6th international joint conference on Autonomous agents and multiagent systems*, 67. ACM.

Vaisbord, I. M., and Zhukovskii, V. I. 1980. Introduction to many player differential games. *Sovetskoe Radio, Moskva*. in Russian.

Vickrey, D., and Koller, D. 2002. Multi-agent algorithms for solving graphical games. In *AAAI/IAAI*, 345–351.

Yokoo, M., and Hirayama, K. 2000. Algorithms for distributed constraint satisfaction: A review. *Autonomous Agents and Multi-Agent Systems* 3(2):185–207.

Yokoo, M.; Ishida, T.; Durfee, E.; and Kuwabara, K. 1992. Distributed constraint satisfaction for formalizing distributed problem solving. In *Distributed Computing Systems, 1992., Proceedings of the 12th International Conference on*, 614–621.

Zhukovskii, V., and Chikrii, A. A. 1994. Linear quadratic differential games. *Naoukova Doumka, Kiev*.