

A combinatorial min-max theorem and minimization of pure-Horn functions*

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Introduction

A Boolean function of n variables is a mapping from $\{0, 1\}^n$ to $\{0, 1\}$. Boolean functions naturally appear in many areas of mathematics and computer science and constitute a key concept in complexity theory. In this paper we shall study an important problem connected to Boolean functions, a so called Boolean minimization problem, which aims at finding a shortest possible representation of a given Boolean function. The formal statement of the Boolean minimization problem (BM) of course depends on how the input function is represented, and how the size of the output is measured.

One of the most common representations of Boolean functions are conjunctive normal forms (CNFs). There are two usual ways how to measure the size of a CNF: the number of clauses and the total number of literals (sum of clause lengths). It is easy to see that BM is NP-hard if both input and output is a CNF (for both measures of the size of the output CNF). This is an easy consequence of the fact that BM contains the CNF satisfiability problem (SAT) as its special case (an unsatisfiable formula can be trivially recognized from its shortest CNF representation). In fact, BM was shown to be in this case probably harder than SAT: while SAT is NP-complete (i.e. Σ_1^P -complete (Cook 1971)), BM is Σ_2^P -complete (Umans 2001) (see also the review paper (Umans, Villa, and Sangiovanni-Vincentelli 2006) for related results). It was also shown that BM is Σ_2^P -complete when considering Boolean functions represented by general formulas of constant depth as both the input and output for BM (Buchfuhrer and Umans 2011).

Due to the above intractability result, it is reasonable to

study BM for subclasses of Boolean functions for which SAT (or more generally consistency testing, if the function is not represented by a CNF) is solvable in polynomial time. A good example of such a class is the class of Horn functions. A CNF is Horn if every clause in it contains at most one positive literal, and it is pure Horn (or definite Horn in some literature) if every clause in it contains exactly one positive literal. A Boolean function is (pure) Horn, if it admits a (pure) Horn CNF representation. Pure Horn functions represent a very interesting concept which was extensively studied in many areas of computer science and mathematics. This concept appears as directed hypergraphs in graph theory and combinatorics, as implicational systems in artificial intelligence and database theory, and as lattices and closure systems in algebra. This identical concept has traditionally been studied within logic, combinatorics, database theory, artificial intelligence, and algebra using different techniques, different terminology, and often exploring similar questions with somewhat different emphasis corresponding to the particular area. Nevertheless, in each of these areas the problem equivalent to BM, i.e. a problem of finding the shortest representation was studied. For instance, one of the basic results, the existence of the GD-basis, was discovered independently and has different proofs in several of the above mentioned areas.

There are several ways how to measure the size of pure Horn function representation by a formula (or by a directed hypergraph or by an implicational system). Five different measures were introduced in (Ausiello, D'Atri, and Sacca 1986), including the most common one (the number of clauses which is the same as the number of hyperarcs in the hypergraph context). For four of these measures it is NP-hard to find the shortest representation, the sole exception being the number of bodies (number of source sets in the hypergraph context and the number of rules in the implicational system context), for which a polynomial time procedure exists to derive a minimum representation. The first such algorithm appeared in database theory literature (Maier 1980). Different algorithms for the same task were then independently discovered in hypergraph theory (Ausiello, D'Atri, and Sacca 1986), and in the theory of closure systems (Guigues and Duquenne 1986).

It may be somewhat puzzling what makes the last measure so different (in terms of tractability of minimization)

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from the other four. In this paper, we will try to provide one possible explanation. While there is a gap between the smallest number of clauses in any representation of a given pure Horn function and a natural lower bound on this number (Boros et al. 2010; Čepek, Kučera, and Savický 2012; Hellerstein and Kletenik 2013) (i.e. only a weak duality between upper and lower bounds exists), we shall show here a strong duality between the smallest number of source sets in any representation of a given pure Horn function and the corresponding lower bound on this number.

There is an extensive literature studying Horn minimization, that is the Boolean minimization problem where the input is some representation of a Horn function. When the objective is to minimize the number of clauses (hyperarcs), the problem is NP-hard, as was first observed in (Ausiello, D’Atri, and Sacca 1986) and later independently in (Hammer and Kogan 1993). Both proofs construct high degree clauses (with the degree proportional to the number of all variables, where the degree of a clause is the number of literals in it), which left open the question, what is the complexity of clause minimization for pure Horn CNFs of a bounded degree. It can be shown, that clause minimization stays NP-hard even when the inputs are limited to cubic (degree at most three) pure Horn CNFs (Boros, Čepek, and Kogan 1998). It should be also noted, that there exists a hierarchy of tractable subclasses of Horn CNFs for which there are polynomial time algorithms minimizing the number of clauses, namely acyclic and quasi-acyclic Horn CNFs (Hammer and Kogan 1995), and CQ Horn CNFs (Boros et al. 2009). There are also few heuristic minimization algorithms for Horn CNFs (Boros, Čepek, and Kogan 1998).

Recently, it was shown in (Bhattacharya et al. 2010; Boros and Gruber 2014) that pure Horn minimization is not only hard to solve exactly but even hard to approximate. More precisely, (Bhattacharya et al. 2010) shows that this problem is inapproximable within a factor $2^{\log^{1-\varepsilon}(n)}$ assuming $NP \not\subseteq DTIME(n^{poly\log(n)})$, and (Boros and Gruber 2014) that it is inapproximable within a factor $2^{O(\log^{1-o(1)} n)}$ assuming $P \not\subseteq NP$ even when the input is restricted to 3-CNFs with $O(n^{1+\varepsilon})$ clauses, for some small $\varepsilon > 0$. The latter result of course implies, that pure Horn minimization (both with respect to the number of clauses and the number of literals) is NP-hard already for cubic CNFs.

In this paper we focus on pure Horn functions and on the minimization of the number of its source sets. We derive a new min-max relation that provide a new certificate for optimality. We also provide a polynomial time algorithm to derive a unique source minimal form (also known as the GD-basis of the underlying pure Horn function, see (Guigues and Duquenne 1986)) that reveals more structure then previous algorithms. IN particular, it shows that any pure Horn function h has a unique integer $k^* = k^*(h)$ and unique pure Horn majorants $1 = h_0 \geq h_1 \geq \dots \geq h^{k^*} = h$ that can be obtained by our algorithm from any CNF representation of h such that for any integer $0 < i < k^*$ we also have $k^*(h_i) = i$, and obtain h_1, \dots, h_{i-1} as its majorants from any CNF representation of h_i . Furthermore, we derive some ad-

ditional properties of source-minimal CNF expressions, that shed new light on the hard problem of clause minimization and results in a potential decomposition of this hard problem into smaller independent clause minimization problems.

For the sake of brevity we leave out most proofs for this preliminary version.

Definitions

We denote by V the set of variables, set $n = |V|$, and consider Boolean functions $f : \mathbb{B}^V \rightarrow \mathbb{B}$, where $\mathbb{B} = \{0, 1\}$. We shall write $f \leq g$ if for all $X \in \mathbb{B}^V$ we have $f(X) \leq g(X)$. We denote by $\mathbb{T}(f) = \{X \in \mathbb{B}^V \mid f(X) = 1\}$ the set of *true points* of f , and by $\mathbb{F}(f) = \mathbb{B}^V \setminus \mathbb{T}(f)$ its set of *false points*. Note that there is a one-to-one correspondence between binary vectors and subsets. Namely, to a subset $S \subseteq V$ we can associate its *characteristic vector* $\chi_S \in \mathbb{B}^V$, while to a binary vector $X \in \mathbb{B}^V$ we can associate its *support* $ON(X) = \{i \in V \mid x_i = 1\} \subseteq V$. Thus we have $ON(\chi_S) = S$ for all $S \subseteq V$.

The components $x_i, i = 1, \dots, n$ can be viewed as Boolean variables (where truth values are represented by 0 and 1). The logical negation of these variables will be denoted by $\bar{x}_i = 1 - x_i, i = 1, \dots, n$, and called *complemented variables*. Since variables and their complements frequently play a very symmetric role, we call them together as *literals*, and introduce $\mathbf{L} = \{x_1, \bar{x}_1, x_2, \bar{x}_2, \dots, x_n, \bar{x}_n\}$.

It is well known that every Boolean function can be represented by a CNF. A clause (an elementary disjunction of literals) is Horn if at most one of its literals is an uncomplemented variable. A Boolean function is Horn, if it can be represented by a Horn CNF, that is a conjunction of Horn clauses. Implicates, prime implicates, pure-Horn, etc.

We consider pure-Horn functions f in CNF representation: $\mathcal{I}(f)$ denotes the set of Horn implicates of f , $\mathcal{P}(f) \subseteq \mathcal{I}(f)$ denotes the set of prime implicates of f , and $\mathcal{P}^*(f) \subseteq \mathcal{I}(f)$ is the resolution closure of $\mathcal{P}(f)$. In the sequel we assume that we have a pure-Horn function f given, and will relate all other definitions to it.

For a subset S of the variables V we denote by $F_f(S)$ and $F_\Phi(S)$ the forward chaining closure of S , with respect to the function f or DNF Φ .

For a subset S and variable $u \notin S$ we write $S \rightarrow u$ to denote the pure Horn clause $C = u \vee \bigvee_{v \in S} \bar{v}$, where S is called its body (or source set) and u is its head. For two subsets A, B of the variables we write $A \rightarrow B$ to denote the conjunction (or set) of the clauses

$$\bigwedge_{u \in B} (A \rightarrow u).$$

For a subset $\Phi \subseteq \mathcal{P}^*(f)$ we shall view Φ both as a set and as a conjunction of clauses. We also interpret any subset $\Phi \subseteq \mathcal{P}^*(f)$ as a function (represented by the corresponding CNF.) Furthermore, by writing $\Phi = f$, $\Phi = \Psi$ and $\Phi \neq \Psi$ we mean that Φ represents the same function as f and Ψ , and that it does not represent the same function as Ψ . This will never cause confusion, since we do not need to compare in the sequel by equality/non-equality subsets of implicates, as

set families. We shall write $\Psi \subseteq \Phi$ if Ψ , as a set of clauses, is a subset of Φ . We shall write $\Psi \leq \Phi$ if the Boolean functions defined by these conjunctions have this relation, that is if $\Psi(X) \leq \Phi(X)$ for all $X \in \mathbb{B}^V$.

Remark 0.1. *It is easy to see that the forward chaining operator satisfies the following properties. Assume that $\Psi \subseteq \Phi$ and $A \subseteq B \subseteq V$.*

$$F_\Phi(A) = F_\Phi(F_\Phi(A)) \quad (1a)$$

$$F_\Phi(A) \subseteq F_\Phi(B) \quad (1b)$$

$$F_\Psi(A) \subseteq F_\Phi(A) \quad (1c)$$

Remark 0.2. *If f is a pure-Horn function and $\Phi \subseteq \mathcal{P}^*(f)$, then $f = \Phi$ iff $F_f(S) = F_\Phi(S)$ for all subsets $S \subseteq V$. Furthermore $S \rightarrow u$ is an implicate of f (that is $u \vee \bigvee_{v \in S} \bar{v} \geq f$) iff $u \in F_f(S)$.*

For an implicate $C \geq f$ we denote by $body(C)$ its body (as a set of variables), and by $head(C)$ its head (as a single variable.)

For a subset $S \subseteq V$ of the variables we denote by $\mathcal{C}_S = S \rightarrow (F_f(S) \setminus S)$ the set of implicates of f with body S . For a CNF $\Phi = f$ representing the pure-Horn function f we denote by Φ_S the set of clauses C of Φ with $body(C) = S$.

A subset $\Phi \subseteq \mathcal{P}^*(f)$ is called *right-saturated*, if for every clause $(S \rightarrow u) \in \Phi$ we have $\mathcal{C}_S \subseteq \Phi$. A CNF Φ is called *body-irredundant* if for every $\emptyset \neq \Phi_S$ the CNF (set of clauses) $\Phi \setminus \Phi_S$ represents a function different from Φ .

A subset $\emptyset \neq \mathcal{E} \subseteq \mathcal{P}^*(f)$ is called *essential*, if whenever the resolution of two clauses C_1 and C_2 belongs to \mathcal{E} , at least one of C_1 and C_2 must also belong to \mathcal{E} .

Remark 0.3. *A subset $\Phi \subseteq \mathcal{P}^*(f)$ represents f iff $\Phi \cap \mathcal{E} \neq \emptyset$ for all essential sets \mathcal{E} .*

To a subset $S \subseteq V$ of the variables we associate the following subset of implicates

$$\mathcal{E}_S = \{C \geq f \mid body(C) \subseteq S, head(C) \notin S\}. \quad (2)$$

Remark 0.4 ((Cepk, Kučera, and Savický 2012)). *If $\mathcal{E}_S \neq \emptyset$, then \mathcal{E}_S is an essential set. Furthermore, for every essential set \mathcal{E} there exists a subset $S \subseteq V$ such that $\emptyset \neq \mathcal{E}_S \subseteq \mathcal{E}$.*

For a CNF $\Phi \subseteq \mathcal{P}^*(f)$ we denote by $\mathcal{B}(\Phi) = \{S \subseteq V \mid \exists u \in V \text{ s.t. } S \rightarrow u \in \Phi\}$ its set of bodies.

Let us introduce two measures for the size of a representation of f :

$$cnf(f) = \min_{\substack{\Phi \subseteq \mathcal{P}^*(f) \\ \Phi = f}} |\Phi|$$

and

$$body(f) = \min_{\substack{\Phi \subseteq \mathcal{P}^*(f) \\ \Phi = f}} |\mathcal{B}(\Phi)|$$

Remark 0.5. *By definition we have*

$$body(f) \leq cnf(f).$$

Let us define $ess(f)$ as the maximum cardinality of a family of pairwise disjoint essential sets.

Let us further call two essential sets \mathcal{E} and \mathcal{E}' *body-disjoint* if there are no implicates $S \rightarrow u, S \rightarrow v \in \mathcal{P}^*(f)$

such that $S \rightarrow u \in \mathcal{E}$ and $S \rightarrow v \in \mathcal{E}'$. Clearly, body-disjoint essential sets are also disjoint, since $u = v$ is possible in the above definition. We define $bess(f)$ as the maximum cardinality of a family of pairwise body-disjoint essential sets.

Remark 0.6. *By Remarks 0.3 and 0.4 and by the above definitions we have*

$$bess(f) \leq ess(f) \leq cnf(f).$$

Let us finally remark, that disjointness or body-disjointness of essential sets of the form (2) can be tested efficiently.

Remark 0.7. *Given subsets $P, Q \subseteq V$, the essential sets \mathcal{E}_P and \mathcal{E}_Q are disjoint iff $F_f(P \cap Q) \subseteq P \cup Q$. They are body-disjoint iff either $F_f(P \cap Q) \subseteq P$ or $F_f(P \cap Q) \subseteq Q$. Thus, both properties can be tested in polynomial time in terms of the size of a pure Horn CNF representing f .*

As a consequence, lower bounds on the quantities $ess(f)$ and $bess(f)$ have polynomial certificates. For instance, to prove that $K \leq bess(f)$ for a pure Horn function f represented by a pure Horn CNF Φ , it is enough to exhibit subsets $Q_i, i = 1, \dots, K$ such that the essential sets $\mathcal{E}_{Q_i}, i = 1, \dots, K$ are pairwise body-disjoint. By Remark 0.7 the latter can be verified in polynomial time in terms of K and the size of Φ .

Strong Duality

Our main result in this section is the min-max theorem claiming that the maximum number of pairwise body-disjoint essential sets is the same as the minimum number of bodies one needs in a representation of the function.

Let us show first a weak dual relation between these quantities.

Lemma 0.8 (Weak Duality). *Let f be a pure Horn function, $\mathcal{E}_i \subseteq \mathcal{P}^*(f)$, $i = 1, \dots, m$ be an arbitrary family of pairwise body-disjoint essential sets, and let $\Phi \subseteq \mathcal{P}^*(f)$ be an arbitrary CNF representation of f . Then*

$$m \leq |\mathcal{B}(\Phi)|.$$

Consequently, we have

$$bess(f) \leq body(f).$$

Proof. Since these essential sets are pairwise body-disjoint, the sets $\mathcal{B}(\Phi \cap \mathcal{E}_i) \subseteq \mathcal{B}(\Phi)$, $i = 1, \dots, m$ are also pairwise disjoint nonempty sets, implying the first claim. Applying it to a maximum cardinality family of pairwise body-disjoint essential sets, and a CNF of f with the minimum number of bodies in it, we obtain the second inequality. \square

Lemma 0.9. *Let f be an arbitrary pure Horn function, set $\mathcal{B}(\Phi^*) = \{S_1, \dots, S_m\}$, denote by $\Phi_{-i} = \Phi^* \setminus \mathcal{C}_{S_i}$ the truncated CNF obtained by removing all clauses from Φ^* with body S_i , and define*

$$P_i = F_{\Phi_{-i}}(S_i)$$

as the forward chaining closure of S_i with respect to the above defined truncated CNF Φ_{-i} , $i = 1, \dots, m$. Then, the essential sets \mathcal{E}_{P_i} , $i = 1, \dots, m$ are pairwise body disjoint and nonempty.

We leave out the proof of this claim for the sake of brevity.

Corollary 0.10. *If Φ^* is a body-irredundant right-saturated CNF of a pure Horn function f , then*

$$bess(f) \geq |\mathcal{B}(\Phi^*)|.$$

Proof. Follows by Lemma 0.9. □

Theorem 0.11 (Strong Duality). *Let f be an arbitrary pure Horn function. Then, we have*

$$bess(f) = body(f).$$

Furthermore, any body-irredundant right-saturated CNF of f is body minimum.

Proof. Consider an arbitrary body minimum pure Horn CNF Φ representing f , that is with $|\mathcal{B}(\Phi)| = body(f)$. Let us add all necessary clauses to make it right-saturated, and denote the obtained CNF by Φ^* . We have $\mathcal{B}(\Phi^*) = \mathcal{B}(\Phi)$. Since Φ is body minimum, both Φ and Φ^* must be body-irredundant. Thus, we can apply Corollary 0.10, and obtain

$$bess(f) \geq |\mathcal{B}(\Phi^*)| = |\mathcal{B}(\Phi)| = body(f).$$

By Lemma 0.8 we have

$$body(f) \geq bess(f).$$

Thus, $bess(f) = body(f)$ is implied. Furthermore, by Corollary 0.10 and by the definition of $body(f)$ we have

$$bess(f) \geq |\mathcal{B}(\Phi^{**})| \geq body(f)$$

for an arbitrary body-irredundant right-saturated CNF Φ^{**} , implying by the above equality that Φ^{**} is body minimum. □

Corollary 0.12. *For a pure Horn function f we have*

$$bess(f) \leq cnf(f) \leq n \cdot bess(f).$$

Proof. Consider an arbitrary irredundant Horn CNF Φ^* of f . Then by Theorem 0.11 we have

$$bess(f) = body(f) = |\mathcal{B}(\Phi^*)| \leq cnf(f) \leq |\Phi^*| \leq n \cdot |\mathcal{B}(\Phi^*)| = n \cdot bess(f). □$$

The example

$$f(x_1, \dots, x_n) = \bigwedge_{i=1}^n x_i$$

shows that the above inequalities are best possible, since here we have $bess(f) = 1$ and $cnf(f) = n$.

Examples for non-uniqueness

Let us consider the following example, defining a pure Horn function h in four variables. To simplify notation, we use only the indices of these variables, and e.g., 12 means the set of the first two variables.

$$\begin{aligned} 1 &\rightarrow 2 \\ 2 &\rightarrow 1 \\ 13 &\rightarrow 24 \\ 23 &\rightarrow 14 \end{aligned}$$

It is easy to see that $body(h) = bess(h) = 3$, and in fact both of the following CNF-s are body-irredundant and right-saturated (that is body minimum representations of h .)

$$\begin{array}{ll} 1 \rightarrow 2 & 1 \rightarrow 2 \\ 2 \rightarrow 1 & 2 \rightarrow 1 \\ 13 \rightarrow 24 & 23 \rightarrow 14 \end{array}$$

Thus, body minimum representations of a pure Horn function are not unique.

A polynomial algorithm to produce a (unique) body minimal representation

In the rest of the paper we show that every pure Horn function has a unique irredundant and saturated (and hence body minimum) representation that fulfills an additional property. This unique body minimum representation can be obtained in polynomial time from any input CNF of the Horn function.

In fact the unique CNF representations obtained by our algorithm correspond to the so called GD-bases (see (Guigues and Duquenne 1986)) of implication systems (which are equivalent to the pure Horn CNF-s, as we emphasized from the very beginning.) Maier (Maier 1980) proved that a body minimum form can be obtained in polynomial time, in the context of functional dependencies, which are in fact implication systems. Our algorithm here is different from the one suggested by Maier, and its correctness and running time analysis is quite simple, due to the fact that we use that every implication system (pure Horn CNF) defines a unique pure Horn function, and true/false assignments of this functions have direct relations to the properties of subfamilies of its implicates.

Our algorithm also shows that every pure Horn function h defines a finite hierarchy of upper approximating pure Horn functions

$$h = h^k \leq h^{k-1} \dots \leq h^1 \leq h^0 = 1$$

where k and the pure Horn functions h^i , $i = 0, \dots, k$ are determined uniquely by h . The unique GD-basis for each of these functions can be obtained, as a side product of our algorithm, in polynomial time, from any pure Horn CNF representation of h .

Given a pure Horn function h , an assignment $x \in \{0, 1\}^V$ is a *true point* of h if $h(x) = 1$. We denote by $\mathbb{T}(h) \subseteq \{0, 1\}^V$ the set of true points of h . For a vector $x \in \{0, 1\}^V$ we denote by $ON(x)$ the set of variables which are assigned value 1 in x , and denote by $OFF(x)$ the set of variables that receive value 0 in assignment x .

Remark 0.13. Given a pure Horn CNF Φ , representing the pure Horn function h , we have

$$x \in \mathbb{T}(h) \iff F_\Phi(ON(x)) = ON(x).$$

Given a hypergraph $\mathcal{S} \subseteq 2^V$ (of subsets of variables), we denote by $\min^1 \mathcal{S}$ the subfamily consisting of all containmentwise minimal sets of \mathcal{S} . For instance, if $\mathcal{S} = \{1, 2, 14, 23\}$, then $\min^1 \mathcal{S} = \{1, 2\}$.

Now we are ready to describe a simple and efficient algorithm that produces a unique body minimal representation from an arbitrary Horn CNF.

Algorithm 1 BODYMIN(Φ)

Require: a pure Horn CNF Φ representing the pure Horn function h .

Ensure: a unique body minimal representation Ψ of h .

- 1: Initialize $i = 0$, $\mathcal{S}^0 = \mathcal{B}(\Phi)$, $\Psi^0 = 1$, $\mathcal{T}^0 = \emptyset$.
 - 2: **while** $\mathcal{S}^i \neq \emptyset$ **do**
 - 3: $\mathcal{S}^{i+1} = \{F_{\Psi^i}(S) \mid S \in \mathcal{S}^0, F_h(S) \supsetneq F_{\Psi^i}(S)\}$
 - 4: $\mathcal{T}^{i+1} = \min^1 \mathcal{S}^{i+1}$
 - 5: $\Psi^{i+1} = \Psi^i \wedge \bigwedge_{T \in \mathcal{T}^{i+1}} (T \rightarrow (F_h(T) \setminus T))$
 - 6: $i \leftarrow i + 1$
 - 7: **end while**
 - 8: $k^* = i$ and $\Psi = \Psi^{k^*}$
-

Theorem 0.14. Let h be a pure Horn function. Then, algorithm *BodyMIN*(Φ) outputs in polynomial time a **unique** body minimal (irredundant and saturated) CNF Ψ of h , for all CNF-s Φ representing h .

We leave out the proof of this claim for the sake of brevity.

Let us recall that by (Guigues and Duquenne 1986) every implication system has a unique implication-minimum representation, its so called GD-basis. Equivalently, every pure Horn function has a unique body-minimum CNF representation, which we can also call its GD-basis. Let us show next that Ψ^{k^*} produced by BODYMIN(Φ) is this GD-basis of the underlying pure Horn function h .

To do so we need to introduce the notion of *left-saturation*.

Definition 0.15. A CNF $\Phi = \bigwedge_{i=1}^m A_i \rightarrow B_i$ is left-saturated if for all $i \neq j$ the relation $A_i \subseteq A_j$ implies $B_i \subseteq A_j$.

It is easy to see that from any CNF $\Phi = \bigwedge_{i=1}^m A_i \rightarrow B_i$ representing the pure Horn function h we can obtain an equivalent CNF Φ^* of h that is left-saturated by the following iterative steps.

If there are indices $i \neq j$ such that $A_i \subseteq A_j$ and variable $v \in B_i \setminus A_j$ we update Φ as

$$\Phi' = ((A_j \cup \{v\}) \rightarrow B_j) \wedge \bigwedge_{\substack{i=1 \\ i \neq j}}^m A_i \rightarrow B_i.$$

It is easy to see that Φ' is also a CNF of h . Furthermore, the above steps can only be performed finitely many times, and hence we obtain at the end a left-saturated CNF Φ^* of h . On the surface this procedure may produce different left-saturated CNF-s, depending on the order we pick the indices. In fact this is not the case, under some mild conditions, as we shall show shortly. Let us note first that uniqueness certainly may not hold if we have $A_i = A_j$ for some $i \neq j$. For instance, if $\Phi = (1 \rightarrow 23) \wedge (1 \rightarrow 4)$, then we can arrive to either $\Phi^* = (14 \rightarrow 23) \wedge (1 \rightarrow 4)$ or $\Phi^{**} = (1 \rightarrow 23) \wedge (123 \rightarrow 4)$. To avoid such increase in the number of bodies, we need to assume, as we did in the past, that we have one implication for every body of Φ , that is that $A_i \neq A_j$ whenever $i \neq j$.

Let us also remark that there are several other definitions for left-saturated forms in the literature. For instance in the context of the closure system associated with h , the bodies of a left-saturated CNF are also called pseudo-closed (see e.g. (Wild 1994)).

There is also another left-closure operator considered e.g., in (Wild 1994; Arias and Balcázar 2009; 2011). Given a CNF Φ of h , as above, let us call two bodies A_i and A_j equivalent if $F_\Phi(A_i) = F_\Phi(A_j)$, that is they have the same forward chaining closure by h . Let us denote by $\Phi[A_i]$ the sub-CNF of Φ consisting all implications the body of which are equivalent in this sense with A_i . Then define for every index i the left-closure of body A_i as $A_i^\bullet = F_{\Phi \setminus \Phi[A_i]}(A_i)$, that is as the forward chaining closure of A_i with respect to the implications of Φ that have bodies not equivalent with A_i .

One has to be a bit careful however by interchanging these definitions and operators. Let us consider the following example.

Φ	Φ^\bullet	Φ°	Φ^{\circledast}
$12 \rightarrow 5$	$12346 \rightarrow 5$	$123467 \rightarrow 5$	
$23 \rightarrow 67$	$23 \rightarrow 67$	$23 \rightarrow 67$	$23 \rightarrow 14567$
$1 \rightarrow 46$	$1 \rightarrow 46$	$1 \rightarrow 46$	$1 \rightarrow 346$
$45 \rightarrow 3$	$45 \rightarrow 3$	$45 \rightarrow 3$	$45 \rightarrow 3$
$46 \rightarrow 3$	$46 \rightarrow 3$	$46 \rightarrow 3$	$46 \rightarrow 3$
$67 \rightarrow 1$	$67 \rightarrow 1$	$67 \rightarrow 1$	$67 \rightarrow 134$

In this example the first column represents the clause-irredundant CNF Φ representing a pure Horn function h , the second column shows the results of applying operator $A \rightarrow A^\bullet$ to Φ , the third column is obtained by applying Definition 0.15, while the last column shows the GD-basis of h . In this example 12 and 23 are equivalent bodies (their forward chaining closures include all variables.) Thus, when obtaining $(12)^\bullet$ we could use only the last four implications of Φ . As we can see, Φ^\bullet does not satisfy Definition 0.15,

since $23 \rightarrow 7$ leads out of the body 12346. The reasons for these differences are that Φ is not right-saturated and body-irredundant (and in particular, not body-minimum.) In fact it will follow from some of our claims shown later below that for CNF-s that are body-minimum the above problems and differences will not arise.

Let us also note that if Φ is a right-saturated and body-irredundant CNF, then in Definition 0.15 we could have used \subsetneq instead of \subseteq .

With these notions in mind, let us show next that Ψ^{k^*} is not only right-saturated and body-irredundant (body-minimum) but it is also left-saturated e.g., according to Definition 0.15.

Claim 0.16. *If $T, T' \in \mathcal{B}(\Psi^{k^*})$ are two distinct sets, such that $T \subseteq T'$, then we have $F_h(T) \subsetneq T'$.*

Corollary 0.17. *If Φ is a pure Horn CNF representing the pure Horn function h , then Ψ^{k^*} , the output of $\text{BODYMIN}(\Phi)$, is the GD-basis of h .*

Body minimum forms and clause minimization

In what follows we derive a few additional properties for body-minimum CNF representations, prove the claims we made about left-saturation above, and obtain a decomposition method for finding the cnf-minimum of a give pure Horn function.

We shall consider a CNF Φ of the form

$$\Phi = \bigwedge_{i=1}^m A_i \rightarrow B_i \quad (3)$$

where we assume that $A_i \neq A_j$ for $i \neq j$. We define for $1 \leq j \leq m$

$$\Phi_{-j} = \bigwedge_{\substack{i=1 \\ i \neq j}}^m A_i \rightarrow B_i \quad (4)$$

and

$$\Phi^* = \bigwedge_{i=1}^m A_i \rightarrow F_h(A_i), \quad (5)$$

where h is the pure Horn function represented by the CNF Φ . Let us note that Φ^* is also a CNF of h , and it is right-saturated. Let us also define

$$\Phi^\circ = \bigwedge_{i=1}^m F_{\Phi_{-i}}(A_i) \rightarrow B_i \quad (6)$$

and note that Φ° is a left-saturated CNF. Let us add that when writing Φ_{-j}^* we mean first right-saturating Φ and then deleting the implication with body A_j , that is

$$\Phi_{-j}^* = \bigwedge_{\substack{i=1 \\ i \neq j}}^m A_i \rightarrow F_h(A_i).$$

Let us finally define

$$\Phi^{\otimes} = \bigwedge_{i=1}^m F_{\Phi_{-i}^*}(A_i) \rightarrow F_h(A_i). \quad (7)$$

Lemma 0.18. *If Φ is body-minimum, then $\Phi^{\otimes} = h$.*

Corollary 0.19. *If Φ is body minimum, then Φ^{\otimes} is the GD-basis of h .*

Lemma 0.20. *If Φ is body-minimum, then for all $i = 1, \dots, m$ we have*

$$F_{\Phi_{-i}}(A_i) = F_{\Phi_{-i}^*}(A_i).$$

Lemma 0.21. *If Φ is body-minimum, then $\Phi^\circ = h$.*

Claim 0.22. *Assume that $\Phi = \bigwedge_{i=1}^m A_i \rightarrow B_i$ is a clause-irredundant pure Horn CNF representation of the pure Horn function h , that is not body-minimum, and e.g., $A_1 \rightarrow F_h(A_1)$ is redundant in the right-saturated CNF $\Phi^* = \bigwedge_{i=1}^m A_i \rightarrow F_h(A_i)$. Let $v \in B_1$ be an arbitrary variable. Then*

- (i) *there exists an index $j \neq 1$ such that the clause $A_j \rightarrow v$ is an implicate of h , and*
- (ii) *the CNF*

$$\hat{\Phi} = (A_1 \rightarrow B_1 \setminus \{v\}) \wedge (A_j \rightarrow B_j \cup \{v\}) \wedge \bigwedge_{\substack{2 \leq i \leq m \\ i \neq j}} (A_i \rightarrow B_i)$$

is another pure Horn CNF of h .

Note that the above head switching does not increase the number of clauses and does not change the right-saturated form $\Phi_{-1}^* = \hat{\Phi}_{-1}^*$. Consequently, the implications $A_1 \rightarrow F_h(A_1)$ remain redundant in $\hat{\Phi}^*$. Thus we can repeat the same operation with $\Phi \leftarrow \hat{\Phi}$ and with the other variables in B_1 , until A_1 will disappear from Φ , implying by Theorem 0.11 the following corollary.

Corollary 0.23. *Given an irredundant pure Horn CNF Φ representing the pure Horn function h , we can derive in polynomial time another pure Horn CNF $\hat{\Phi}$ of h such that*

- (a) $|\hat{\Phi}| \leq |\Phi|$,
- (b) $\hat{\Phi}$ is body-minimum, and
- (c) $\mathcal{B}(\hat{\Phi}) \subseteq \mathcal{B}(\Phi)$. □

In particular, if we start with a cnf-minimum expression of h , then we can conclude with the following claim.

Corollary 0.24. *Every pure Horn function h has a pure Horn CNF representation that is both cnf-minimum and body-minimum.* □

Corollary 0.25. *Every pure Horn function h has a cnf-minimum representation that has the same bodies as its GD-basis.*

Proof. Follows by Corollary 0.24, and Lemmas 0.20 and 0.21. □

Based on the above and on our algorithm we show next that the problem of finding a cnf-minimum representation for a given pure Horn CNF Φ , can be decomposed into smaller problems typically.

Let us consider a pure Horn function h and its unique GD-basis Φ^{\otimes} , as produced by algorithm BODYMIN(Φ)

$$\Phi^{\otimes} = \bigwedge_{\ell=1}^{k^*} \bigwedge_{T \in \mathcal{T}^\ell} T \rightarrow F_h(T). \quad (8)$$

According to Corollary 0.25 the problem of finding a cnf-minimum representation of h can equivalently be viewed as the problem of finding subsets $B(T) \subseteq F_h(T)$ for all $T \in \mathcal{B}(\Phi^{\otimes}) = \bigcup_{\ell=1}^{k^*} \mathcal{T}^\ell$ such that the CNF

$$\Psi = \bigwedge_{\ell=1}^{k^*} \bigwedge_{T \in \mathcal{T}^\ell} T \rightarrow B(T). \quad (9)$$

represents h and $\sum_{T \in \mathcal{B}(\Phi^{\otimes})} |B(T)|$ is as small as possible. We shall show below that based on the properties of the GD-basis we showed earlier the above problem can further be decomposed into many smaller, independent subproblems of the same type.

Let us introduce

$$\Pi_\ell = \bigwedge_{T \in \mathcal{T}^\ell} T \rightarrow F_h(T)$$

and denote by g_ℓ the pure Horn function represented by Π_ℓ , for $\ell = 1, 2, \dots, k^*$. Note that we have

$$\Psi^k = \bigwedge_{\ell=1}^k \Pi_\ell$$

for the majorizing sub-CNF-s defined in BODYMIN(Φ).

Lemma 0.26. *Assume that Ψ in (9) represents the pure Horn function h . Then for every $\ell = 1, \dots, k^*$ the CNF*

$$\Sigma_\ell = \bigwedge_{T \in \mathcal{T}^\ell} T \rightarrow B(T)$$

represents g_ℓ .

This immediately imply the following

Corollary 0.27. *If Ψ in (9) is a cnf-minimum representation of h , then Σ_ℓ are cnf-minimum representations of $g_\ell = \Pi_\ell$ for all $\ell = 1, \dots, k^*$. Conversely, if Σ_ℓ are cnf-minimum representations of g_ℓ for all $\ell = 1, \dots, k^*$, then Ψ is a cnf-minimum representation of h . \square*

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